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DEGREE SETS OF k-TREES: SMALL k

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ABSTRACT

A k-tree is a k-uniform hypergraph constructed from a single edge by the successive addition of edges each containing a new vertex and $k-1$ vertices of an existing edge. We show that if D is any finite set of positive integers which includes 1, then D is the set of vertex degrees of some k-tree for $k = 2, 3$, and 4, and that there is precisely one such set, $D = \{1, 4, 6\}$, which is not the set of degrees of any 5-tree. We also show for each $k \geqq 2$ that such a set D is the set of degrees of some k-tree provided only that D contains some element d which satisfies $d \geqq k(k-1) - 2\lfloor k/2 \rfloor + 3$.

1. Introduction

By a *k-tree* we mean a k-uniform hypergraph constructed via the following inductive scheme:

(A) The hypergraph with k vertices and a single edge of cardinality k is a k-tree.

(B) If T is a k-tree, the hypergraph obtained by adding to T a single vertex v and a single edge $\{v, v_1, v_2, \cdots, v_{k-1}\}$, where $\{v_1, v_2, \cdots, v_{k-1}\}$ is a subset of some edge of T, is a k-tree.

A tree in the usual terminology of graph theory is thus a 2-tree by this definition. (The hypergraphs which we call "k-trees" have been called $(k-1)$-trees, as in [1] and [5], and $(k-2, k-1)$-trees, as in [2]. For details of some of the results of these works and additional background, see [4].)

By the *degree* of a vertex v of a k-tree T we mean the number of edges of T containing v. Since every k-tree, $k \geqq 2$, contains a vertex of degree 1, for our purposes, a *degree set* will be a finite set $D = \{d_1, d_2, \cdots, d_m\}$ of integers with $1 = d_1 < d_2 < \cdots < d_m$. We will say that D is *realized* by the k-tree T if D is precisely the set of distinct degrees of vertices of T. A list of the degrees of all of the vertices of T will be called a *degree sequence* of T.

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It was shown by Kapoor, Polimeni and Wall in [6] that each degree set is realized by some 2-tree. In [4] it was shown that if D is a degree set with largest element d_m, then either D is realizable for each $k \geqq d_m$ or D is not realizable for any $k \geqq d_m$. In [3] it was proved that for each $k \geqq 2$ a degree set is realizable by a k-tree provided only that d_m is sufficiently large, and hence that all but finitely many such sets are realizable by k-trees. Here we obtain an explicit expression, namely $k(k-1)-2\lfloor k/2 \rfloor+3$, for the size of d_m which insures that D is realizable by a k-tree. We also show that every degree set is realizable for $k=3$ and $k=4$ and that there is precisely one such set, $\{1,4,6\}$, which is not the degree set of any 5-tree.

By a *k-chain* we mean a k-tree with vertices $v_1, v_2, \cdots, v_n$ and edges the sets $\{v_i, v_{i+1}, \cdots, v_{i+k-1}\}$ for each i, $1 \leqq i \leqq n-k+1$. Thus a k-chain with at least k edges has a degree sequence of the form $1,2,3,\cdots,k-1,k,k,\cdots,k,k,k-1,\cdots,3,2,1$. By a *face* of an edge E of a k-tree T we mean a subset of E of cardinality $k-1$. A set of vertices is a face of T if it is a face of some edge of T. If C is a k-chain with vertices and edges as above, we will use F_i^j, $1 \leqq j \leqq i+k-1$, to denote the face $\{v_i, v_{i+1}, \cdots, v_{i+k-1}\}-\{v_j\}$. We will use the term *leaf* to denote an edge added to a k-tree T and sharing a face with T.

In order to obtain k-trees which realize prescribed degree sets we will make frequent use of the general approach described in [3] which is based on attaching leaves to chains. We will also use the following three facts which were given in [4].

(a) For each $k>2$ the degree set $\{1,2,d_3\}$ can be realized by a k-tree which is a "$(k-2)$-fold cone" over the 2-tree having degree set $\{1,2\}$ and d_3 edges.

(b) For each $k>2$ the degree set $\{1,2,d_3,d_4\}$ can be realized by a k-tree which is a "$(k-2)$-fold cone" over the 2-tree having degree set $\{1,2,d_3\}$ and d_4 edges.

(c) ([4], corollary 11) Let $D=\{d_1,d_2,\cdots,d_m\}$ be a degree set with $1=d_1<d_2<\cdots<d_m$ and suppose that $f_2,f_3,\cdots,f_{m-1}$ are positive integers satisfying $\Sigma_{i=2}^{m-1}(d_m-d_i)f_i=d_m-1$. Then D can be realized by a k-tree for each $k \geqq (\Sigma_{i=2}^{m-1} f_i)+1$. (It was shown in [4] that such a realization can always be found having d_m edges, d_m-1 vertices of degree 1, and f_i vertices of degree d_i for each i, $2 \leqq i \leqq m-1$.)

2. Realizability for $k \leqq 5$

The 2-trees which we would call 2-chains are simply paths in the usual graph theoretic sense. It is clear that a 2-tree realization can be obtained for any given degree set by attaching edges (leaves) to some or all of the vertices of degree 2 of such a chain. Our first result is established by showing that for $k = 3$ and $k = 4$ realizations can be obtained in almost all cases by doing little more than adding leaves to chains.

Throughout this section we will assume that the degree set in question is $D = \{d_1, d_2, \cdots, d_m\}$ with $1 = d_1 < d_2 < \cdots < d_m$, and we will let D' be the set $D \cap \{2, 3, \cdots, k-1\}$.

The degree set $\{1\}$ can obviously be realized for each $k \geqq 2$ by a k-tree having a single edge. It is also easy to see that $\{1, d\}$ can be realized for each $k \geqq 2$ by a k-tree having d edges all sharing one common face. We will therefore assume in what follows that each degree set has at least three elements.

THEOREM 1. *Each degree set is realizable by a k-tree for $k = 3$ and $k = 4$.*

PROOF. We proceed by constructing the required k-trees in every case. The nature of the construction will depend both on k and on the set D'.

The case $k = 3$, $D' = \{2\}$. Here we begin with a 3-chain having $2m$ vertices and degree sequence $1, 2, 3, 3, \cdots, 3, 3, 2, 1$. The pairs of vertices $\{v_i, v_{i+1}\}$, for $i \equiv 1 \pmod 2$, $3 \leqq i \leqq 2m-3$, form $m-2$ mutually disjoint faces of C. All of these vertices have degree 3 in C. Thus we may attach edges to C in such a way that for each element d of D other than 1 and 2 the degrees of the two vertices of some one of these faces are increased to d, while all new vertices have degree 1.

The case $k = 3$, $D' = \varnothing$. If $3 \in D$ and $d_m \geqq 4$ we begin as in the previous case with a 3-chain C having $2m$ vertices, but now we attach one leaf to each of the $m-1$ faces F_i^{i+2}, $i \equiv 0 \pmod 2$, $2 \leqq i \leqq 2m-2$. This yields a 3-tree in which the vertices of the original chain have degrees $1, 3, 4, 4, \cdots, 4, 4, 3, 1$. Since there are $2(m-2)$ vertices of degree 4 and successive pairs of these vertices form faces of C, we can attach additional leaves as in the previous case to realize D.

If $3 \notin D$ we first add leaves to increase *each* degree in C to 4. This is done at one end of the chain by adding two edges at the face F_1^3 and one at F_1^2 and at the other end by adding two at F_{2m-2}^{2m-2} and one at F_{2m-2}^{2m-1}. The degrees of vertices in the middle of the chain are each increased by 1 by the addition of a single leaf at each of the faces F_i^i, $i \equiv 1 \pmod 2$, $3 \leqq i \leqq 2m-5$. Additional leaves are now added to obtain the remaining degrees in D.

Since D is assumed to have at least 3 elements, no further possibilities exist for $k = 3$.

The case $k = 4$, $D' = \{2, 3\}$. Here we need only begin with a 4-chain having $3m-3$ vertices so that there are $3(m-3)$ vertices of degree 4, since this provides

$m-3$ mutually disjoint faces to which we may attach leaves to realize any degrees in D which are greater than 3.

The case $k=4$, $D'=\{2\}$. If $d_m \geqq 6$ we first add a leaf at each of the faces F_1^2 and F_2^1 of a 4-chain C. By attaching an edge to the leaf already added at F_1^2 we can increase the degrees of the first 5 vertices of C to $2,2,6,6,6$ (while introducing only new vertices of degree 1 and 2). A similar construction is carried out at the other end of C. Having chosen the length of C so that the number of vertices still having degree 4 is an appropriate multiple of 3, we can now realize the remaining elements of D by adding leaves at faces of the midsection of C and, if $6 \notin D$, to the two faces which consist of vertices of degree 6.

If $d_m \leqq 5$, then D is one of the sets $\{1,2,4\}$, $\{1,2,5\}$, and $\{1,2,4,5\}$. These are realizable by (a) and (b) above.

The case $k=4$, $D'=\{3\}$. If $d_m \geqq 7$ we attach a leaf to each of F_2^3 and F_3^1 and then add *two* edges to the leaf at F_3^1. This can be done so as to yield degrees $1,3,3,7,7,7$ in C and new vertices of degree 1 and 3. We repeat this construction at the other end of C and then proceed as in the previous cases.

If $d_m = 6$ we first add one edge at each of F_2^3, F_4^4, and F_4^5 to obtain the degrees $1,3,3,6,6,6,6$ in C.

If $d_m \leqq 5$, D is either $\{1,3,4\}$, $\{1,3,5\}$, or $\{1,3,4,5\}$ and each of these is realizable by (c).

The case $k=4$, $D'=\varnothing$. If $d_m \geqq 7$ we first add 3 leaves at F_1^4, 2 at F_1^3 and 1 at F_1^2 of a 4-chain C. The first 4 vertices of C then have degree 7. Next we add one leaf at each F_5^j, $5 \leqq j \leqq 8$, and at each F_9^j, $9 \leqq j \leqq 12$. This yields a total of 12 vertices of degree 7 forming 4 mutually disjoint faces to which additional leaves may be attached. This construction is repeated to form a similar "bed" of leaves at the other end of C, and leaves are attached to the faces in the middle of the chain as needed.

If $d_m < 7$, then D is either $\{1,4,5\}$, $\{1,4,6\}$, or $\{1,4,5,6\}$. The last of these is realizable by (c). To obtain a 4-tree with degree set $\{1,4,5\}$ we start with a single edge E and attach two edges at one face and one at each of the remaining three faces of E. Attaching three edges instead of two at the one face of E yields a 4-tree with degree set $\{1,4,6\}$.

This completes the proof in this case, and hence the proof of the theorem.

We will follow the same general approach to establish that every degree set except one is realizable by a 5-tree. (We will show later that $\{1,4,6\}$ can not be the degree set of any 5-tree.) In most of the cases for which $D' \neq \emptyset$ we first attach leaves to a 5-chain to increase the degrees of the vertices at the ends of the chain to values which are in D. The remaining degrees in D are then created from the midsection of the chain. As an example, when $D' = \{4\}$ we attach leaves at F_2^3, F_2^6, F_4^5, F_7^7, and F_7^8 of the chain. This yields degrees $1,4,4,7,7,7,7,7,7,7,7,5,5,\cdots$ at one end of the chain, which we will represent by "1, 4, 4, (8)7". Since the number of 7's is a multiple of 4, additional leaves can be added to obtain any higher degree. We repeat this construction at the other end of the chain. Since faces in the middle of the chain can be used to create any degrees greater than 4, it follows that D is realizable in this case whenever $d_m \geqq 7$. Henceforth we will indicate such a construction merely by giving the resulting partial degree sequence for the vertices at one end of the chain, from which the leaf construction can easily be recovered.

When $D' = \emptyset$ we will use techniques from [3], which also form the basis of the constructions in Section 3. As before, we may assume that each degree set in question has at least three elements.

THEOREM 2. *Each degree set except* $\{1,4,6\}$ *is realizable by a* 5-*tree.*

PROOF. Again there are a number of cases to be considered.

The case $D' = \{2\}$. The sequence $2,2,(20)9$ is obtained by starting with $2,2,(5)9$; from this we obtain realizations of all degree sets where $d_m \geqq 9$. If $6 \in D$ and $d_m \geqq 7$ we can use $1,2,6,(4)7$. If $d_m = 6$ or $6 \notin D$, the only remaining degree set not realizable by (a) or (b) is $\{1,2,5,7,8\}$. A realization of this set can be constructed from a 5-chain with only 4 edges (and degree sequence $1,2,3,4,4,3,2,1$) by adding three edges at the face F_2^6 and one each at F_2^4 and F_2^5.

The case $D' = \{3\}$. If $d_m \geqq 9$ we proceed as in the previous case, now using the sequence $3,(20)9$. If $d_m = 8$ we use $1,3,(10)8$, and if $d_m = 7$ we use $1,3,3,(5)7$. If $d_m = 6$ and $5 \in D$, we use $1,3,5,(4)6$. The only remaining degree sets are $\{1,3,5\}$ and $\{1,3,6\}$. The first of these is realizable by (c). For $\{1,3,6\}$ we begin with a 5-chain with 4 edges and add an edge at each of the faces F_2^2, F_2^4, and F_3^5.

The case $D' = \{4\}$. As previously mentioned, the sequence $1,4,4,(8)7$ can be used whenever $d_m \geqq 7$. Hence only the sets $\{1,4,5\}$, $\{1,4,6\}$, and $\{1,4,5,6\}$ remain. Realizations of $\{1,4,5\}$ and $\{1,4,5,6\}$ are assured by (c), while $\{1,4,6\}$ is the one exceptional set specified in the theorem.

The case $D' = \{2,3\}$. If $d_m \geqq 9$, we attach leaves at one end of a chain to obtain 2, 2, (5)9 as when $D' = \{2\}$, and at the other end we obtain 3, (20)9 as for $D' = \{3\}$. If $d_m = 8$ we use 1, 2, 3, (5)8 at both ends and if $6 \in D$ we use 2, 3, 3, (5)6. The remaining sets are $\{1,2,3,5\}$ and $\{1,2,3,7\}$ which are realizable by (b), $\{1,2,3\}$ which is realizable by (a), and $\{1,2,3,5,7\}$ for which a realization is obtained by attaching two edges to the face F_2^2 and one to the face F_1^3 of a 5-chain with four edges.

The case $D' = \{2,4\}$. Here if $d_m \geqq 9$ we again build 2, 2, (5)9 at one end of a chain and at the other obtain 1, 4, 4, (8)9 from the construction used to create 1, 4, 4, (8)7 for $D' = \{4\}$. If $7 \in D$ we use 1, 2, 4, (5)7, if $8 \in D$ we use 1, 2, 4, 4, (5)8, and if $6 \in D$ we use 1, 2, 4, 4, (3)6. The remaining sets, $\{1,2,4\}$ and $\{1,2,4,5\}$, are realizable by (a) and (b) respectively.

The case $D' = \{3,4\}$. If $d_m \geqq 7$ and $6 \notin D$ we use 1, 4, 4, (8)7 at one end and at the other one of the sequences 1, 3, 3, (5)7, 1, 3, (10)8, or 3, (20)9 constructed for $D' = \{3\}$. If $6 \in D$, we use 1, 3, 4, (5)6 at each end. This leaves $\{1,3,4\}$ and $\{1,3,4,5\}$, both of which are realizable by (c).

The case $D' = \{2,3,4\}$. Here we need only create any degrees greater than 4 from the midsection of a chain.

The case $D' = \varnothing$. Here we need almost all of the methods which will be used in the next section to obtain our results for arbitrary k. The complete details of these constructions will be given there.

If $d_m \geqq 12$ we first form the sequence (5)11 and from this obtain (10)12. Although 10 is not a multiple of 4, the degrees of the first 10 vertices of the original chain can be increased from 12 to any larger *even* value by adding leaves in blocks of 5. If, however, D contains no even value as large as 12, we must also add leaves to increase the degrees of the midsection of the chain from 5 to 6. It is then possible to obtain vertices of (odd) degree d_m from those of degree 12. If $5 \in D$, this last construction will require the addition of still more leaves to create some new vertices of degree 5. This is done by the "turret" construction described in Section 3. Finally the remaining degrees are obtained by adding edges to the faces whose vertices have degree 5 or 6 in the resulting 5-tree.

If $d_m = 11$ we use the sequence (5)11. The remaining cases are handled as follows:

(i) If $8 \leqq d_m \leqq 10$, we use 1, 6, (4)8, 6 if $6 \in D$, and 1, 7, (4)8 if $7 \in D$. If $\{8,9\} \subset D$ we use 1, 8, (3)9, 8; if $\{9,10\} \subset D$, we use 1, 9, (2)10, (2)9; if $\{8,10\} \subset D$, we use 1, 8, (4)10, 8, (4)10.

(ii) If $8 \leqq d_m \leqq 10$, $6 \notin D$, $7 \notin D$, and D contains only one of 8, 9, and 10, then

D is one of the sets $\{1,5,8\}$, $\{1,5,9\}$, and $\{1,5,10\}$. By adding one edge each at the faces F_2^j, $4 \leqq j \leqq 6$, and F_3^j, $3 \leqq j \leqq 5$, of a 5-chain with only 4 edges we obtain a realization of $\{1,5,8\}$. Since the four vertices of degree 8 in this 5-tree form a face, it is easy to add additional edges to obtain realizations of $\{1,5,9\}$ and $\{1,5,10\}$.

(iii) Finally, if $d_m < 8$, then either $\{5,7\} \subset D$ and we can use $1,5,5,(3)7$, or D is one of $\{1,5,6\}$ and $\{1,6,7\}$. To obtain a realization of $\{1,5,6\}$ we begin with a single edge E and attach two leaves at one face and one leaf at each of the other faces of E. Adding one more edge so that there are two edges at each of two distinct faces of E yields a realization of $\{1, 6, 7\}$.

This completes the proof for $D' = \varnothing$ and hence for the entire theorem.

The next result will complete our analysis for $k \leqq 5$.

THEOREM 3. *The degree set* $\{1,4,6\}$ *cannot be realized by a 5-tree.*

PROOF. Assume for the purpose of arriving at a contradiction that T is a 5-tree with degree set $\{1,4,6\}$ and let T' be the largest subtree of T having only two endedges, i.e., having only two vertices of degree 1.

If T' has only two edges, then all of the edges of T have a common face, limiting T to a degree set of the form $\{1, d\}$.

If T' has three edges, then the remaining edges of T must each share a face with the central edge of T'. Hence the sum of the degrees in T of the vertices in this edge is $5 + 4(e-1)$, where e is the number of edges of T; but since this is an odd number, these degrees cannot all be 4's and 6's.

If T' has four edges, we may assume that T' has vertices $v_1, v_2, \cdots, v_8$, where v_1 and v_8 have degree 1 in T'. Since the removal from T' of the edges containing v_1 and v_8 must yield a 5-tree consisting of just two endedges, there are two vertices of T', say v_2 and v_7, which have degree 2 in T' and which are not contained in any common edge of T' or T. Since T' is maximal, T can be obtained only by adding leaves which do not contain v_1 or v_8 and which contain at most one of v_2 and v_7. At least 4 such leaves must be added to increase the degrees of v_2 and v_7. Since the sum of the degrees in T' is 20, the sum of the degrees of v_3, v_4, v_5, and v_6 in T would then have to be at least $14 + 4 \cdot 3 > 4 \cdot 6$.

Henceforth we assume that T' has at least five edges. The vertices and edges of T' can then be labelled in such a way that

(1) E_1 is one of the two endedges;

(2) E_i shares a face with E_{i-1} for $i > 1$;

(3) $E_1 = \{v_1, v_2, \cdots, v_5\}$, $E_2 = \{v_2, \cdots, v_6\}$, and $E_3 = \{v_3, \cdots, v_7\}$;

(4) v_{i+4} is in E_i for each i.

Under the conditions just described there are just five possibilities for the subtree T'' of T' which consists of vertices $v_1, v_2, \cdots, v_9$ and edges $E_1, E_2, \cdots, E_5$. Their degree sequences are

(A) $1, 2, 3, 4, \underline{5}, \underline{4}, \underline{3}, \underline{2}, \underline{1}$
(B) $1, 2, 3, \underline{5}, \underline{5}, 3, \underline{3}, \underline{2}, \underline{1}$
(C) $1, 2, 3, \underline{5}, \underline{5}, \underline{4}, 2, \underline{2}, \underline{1}$
(D) $1, 2, 4, \underline{5}, \underline{5}, 2, \underline{3}, \underline{2}, \underline{1}$
(E) $1, 2, \underline{5}, \underline{5}, \underline{5}, 2, 2, \underline{2}, \underline{1}$

The underlined numbers indicate the vertices of E_5; these are the only ones whose degrees may be higher in T'. In all cases the degree of v_2 can be larger in T only by virtue of additional edges each sharing a face with E_2, as any alternative contradicts the maximality of T'. In no case can v_2 have degree 6, since this would require that the degree of at least one of the other vertices of E_2 be greater than 6. Thus v_2 must have degree 4 in T and we consider the possible arrangements of the two edges containing v_2 which are not in T'.

In case (A) one of these two edges must contain the face $\{v_2, v_3, v_4, v_6\}$, else v_5 will have degree greater than 6. The other must contain $\{v_2, v_4, v_5, v_6\}$, for otherwise E_3 would contain at least two vertices of degree 6 and v_3, which therefore could not be in additional edges, would have degree 5 in T. This means that v_5 and v_6 already have degree 6, but one of them must either be in E_6 (if it exists) or in some additional edge containing v_8.

In cases (B) and (C) one of the two edges must miss v_4 and the other v_5 if neither is to have degree greater than 6. But then, as before, v_3 will have degree 5 in T.

Again in case (D) the two edges must miss v_4 and v_5 respectively, leaving both with degree 6. As in case (A), however, one of them must also be in E_6 or in an additional edge containing v_8.

Case (E) is the last and easiest, as one of the vertices v_3, v_4, or v_5 must be in both of the two edges, resulting in a degree greater than 6.

We conclude that T does not exist.

3. Realizability for larger k

In this section we find a number $d(k)$ for each $k \geqq 2$ such that if $d_m \geqq d(k)$, then the degree set $D = \{d_1, \cdots, d_m\}$ is realizable by a k-tree.

We begin with the same technique used for the $D' = \varnothing$ case in Theorems 1 and 2: a "bed" of leaves is added to each end of a k-chain so as to increase the degrees of the vertices at the ends of the chain to a common value $a(k)$. These degrees are then brought up to d_m by adding layers of leaves to the beds or to the whole chain. Other large degrees in D are built on faces in the chain's midsection.

The small degrees in D (those in D', plus, in some cases, the degree k) are created by a modification of the turret construction used in [3].

The bed constructions are described by indicating the multiset of faces of the chain to which the leaves are to be attached. Thus, for example, if $k = 4$, then the face set $\{3F_1^4, 2F_1^3, F_1^2\}$ describes the addition of three leaves to the face $\{v_1, v_2, v_3\}$, two leaves to $\{v_1, v_2, v_4\}$ and one to $\{v_1, v_3, v_4\}$.

THEOREM 4. *Let $d(k) = k(k-1) - 2\lfloor k/2 \rfloor + 3$, $k \geqq 2$. Then any degree set $D = \{d_1, d_2, \cdots, d_m\}$ with $1 = d_1 < d_2 < \cdots < d_m$ and $d_m \geqq d(k)$ is realizable by a k-tree.*

PROOF. Suppose first that k is even. We construct a bed on the first $k(k-1)$ vertices of a chain with $2k(k-1) + (m-2)(k-1)$ vertices, via the face set

$$\{(j-1)F_1^j : 2 \leqq j \leqq k\} \cup$$

$$\{(k/2-1)F_i^j : i \equiv 1 \bmod k,\ k < i \leqq (k-1)^2,\ i \leqq j \leqq i+k-1\}$$

which results in $v_1, \cdots, v_{k(k-1)}$ all attaining degree $a(k) = 1 + (k(k-1)/2)$. The further addition of

$$\{(d_m - a(k))F_i^{i+k-1} : i \equiv 1 \bmod (k-1),\ 1 \leqq i \leqq (k-1)^2\}$$

brings the degrees up to d_m. The construction is duplicated at the other end of the chain.

Each degree d in D other than d_1 or d_m is created, as before, on some face $F = \{v'_1, v'_2, \cdots, v'_{k-1}\}$ of the chain's midsection. If $d \geqq k$ we merely add $d-k$ leaves to F. If $d < k$, i.e. $d \in D'$, a more complicated device is needed. Let $p = (k-1)/\text{g.c.d.}(k-1, d-1)$ and add p new vertices $v''_1, \cdots, v''_p$ with corresponding leaves $\{v''_i, v'_1, v'_2, \cdots, v'_{k-1}\}$. Let v^j_i, $1 \leqq i \leqq p$, $1 \leqq j \leqq d-1$ be additional new vertices and include each in an edge of the form

$$\{v^j_i, v''_i, v'_1, v'_2, \cdots, v'_{k-1}\} - \{v'_c\}$$

where $c \equiv ip + j \bmod (k-1)$ with $1 \leqq c \leqq k-1$. We then have that the degree of each v^j_i is 1, the degree of each v''_i is $1 + (d-1) = d$, and the degree of v'_i is $k + p + ((k-2)/(k-1))p(d-1)$. The worst case (highest value of the degree of v'_i) occurs when $d = k-1$, when that value is $k + (k-1) + (k-2)((k-1)-1) = k^2 - 2k + 3 = d(k) \leqq d_m$. We bring the degree of each v'_i up to d_m by adding more leaves to F. This completes the proof when k is even.

If k is odd we start with a chain of $k(k-1)+(m-2)(k-1)$ vertices and build a bed on the first (and last) $k(k-1)/2$ vertices. When $k \equiv 3 \bmod 4$ the face set is

$$\{(j-1)F_1^j : 2 \leqq j \leqq k\} \cup$$

$$\{((k-3)/2)F_i^j : i \equiv 1 \bmod k,\ k < i \leqq (k-2)(k-1)/2,\ i \leqq j \leqq i+k-1\} \cup$$

$$\{F_i^j : i \equiv 1 \bmod 2k,\ 2k < i \leqq (k-2)(k-1)/2,\ i \leqq j \leqq i+((k-1)/2)\} \cup$$

$$\{F_i^j : i \equiv 1+k \bmod 2k,\ k < i \leqq (k-2)(k-1)/2,\ i+((k-1)/2) < j \leqq i+k-1\} \cup$$

$$\{F_i^j : i \equiv 1\ (1+3k)/2 \bmod 2k,\ 1 < i \leqq (k-2)(k-1)/2\}$$

producing degree $a(k) = 1+(k(k-1)/2)$ for vertices $v_1, \cdots, v_{k(k-1)/2}$; if $k \equiv 1 \bmod 4$, the face set is

$$\{(j-1)F_1^j : 2 \leqq j \leqq k\} \cup$$

$$\{((k-3)/2)F_i^j : i \equiv 1 \bmod k,\ k < i \leqq (k-2)(k-1)/2,\ i \leqq j \leqq i+k-1\} \cup$$

$$\{F_i^j : i \equiv 1 \bmod 2k,\ 2k < i \leqq (k-2)(k-1)/2,\ i \leqq j < i+((k-1)/2)\} \cup$$

$$\{F_i^j : i \equiv 1+k \bmod 2k,\ k < i \leqq 1+(k(k-7)/2),\ i+((k-1)/2) < j \leqq i+k-1\} \cup$$

$$\{F_i^j : i \equiv (1+3k)/2 \bmod 2k,\ 1 < i \leqq (k-2)(k-1)/2\} \cup$$

$$\{F_i^{i+k-1} : i \equiv 1 \bmod (k-1),\ 1 \leqq i \leqq (k-2)(k-1)/2\} \cup$$

$$\{F_i^j : i = (k-2)(k-1)/2,\ i \leqq j \leqq i+((k-1)/2)\}$$

resulting in the common degree $a(k) = 2+(k(k-1)/2)$.

When k is odd and $d_m > 1+a(k)$ we pad the beds further with

$$\{\lfloor (d_m - a(k))/2 \rfloor F_i^{i+k-1} : i \equiv 1 \bmod (k-2),\ 1 \leqq i \leqq (k-2)(k-1)/2\} \cup$$

$$\{\lfloor (d_m - a(k))/2 \rfloor F_i^j : i \equiv 1 \bmod k,\ 1 \leqq i \leqq (k-2)(k-1)/2,$$
$$j \equiv 1 \bmod (k-2),\ i \leqq j \leqq i+k-1\}.$$

If $d_m - a(k)$ is even, this brings the degrees of the vertices at the ends of the chain up to d_m, and the rest proceeds exactly as in the case for k even.

When $d_m - a(k)$ is odd it becomes necessary to pad the entire chain with a layer of leaves, namely

$$\{F_i^{i+k-1} : i \equiv 1 \bmod (k-1),\ 1 \leqq i \leqq k(k-1)+(m-2)(k-1)\}$$

resulting in degree d_m for the vertices at the ends of the chain, but pushing the degree of the vertices in the midsection to $k+1$. We will now need a turret to recreate the degree k if $k \in D$, but in this case $p = 1$ and for this we need only $d_m \geqq 2k$. Thus the worst case is still $d = k - 1 \in D$, which here requires $d_m \geqq k^2 - 2k + 4$.

Combining the cases for even and odd k now yields the statement of the theorem.

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